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Prequantizable Poisson manifolds and Jacobi structures

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Abstract. We prove that the total space of a prequantization bundle of a prequantizable Poisson manifold admits a Jacobi structure and, conversely, a regular Jacobi manifold is the prequantization bundle of a Poisson manifold. Illustrative examples are presented.

1. Introduction

Jacobi structures were introduced by Lichnerowicz [13, 14] and they are an important setting for physics. In fact, the algebra of functions $C^\infty(M, \mathbb{R})$ of a Jacobi manifold M is endowed with a Jacobi bracket in such a way that it is a local Lie algebra in the sense of Kirillov ([10]). Jacobi structures are natural generalizations of symplectic and Poisson manifolds.

The first step for the geometric quantization theory due to Konstant and Souriau is to find a prequantization bundle, that is, a principal circle bundle over the symplectic manifold. Thus, the obstruction to the existence of a prequantization is just the de Rham cohomology class of the symplectic form. For Poisson manifolds, the obstruction is the Lichnerowicz–Poisson cohomology class of the 2-vector (see [20]).

The aim of this paper is to prove that the total space of a prequantization bundle of a prequantizable Poisson manifold admits a Jacobi structure. Conversely, we also prove that a regular Jacobi manifold endowed with an appropiate 1-form, is the prequantization bundle of a Poisson manifold. Roughly speaking, we can say that prequantization bundles of Poisson manifolds are just regular Jacobi structures. We analyse three particular cases: Lie–Poisson structures, symplectic manifolds and cosymplectic manifolds. The corresponding Jacobi structures are linear structures, contact or locally conformal symplectic manifolds, respectively.

2. Jacobi and Poisson manifolds

2.1. Prequantizable Poisson manifolds and the Lichnerowicz–Poisson cohomology

Let $\bar{M}$ be a C^∞ manifold. Denote by $\mathfrak{X}(\bar{M})$ the Lie algebra of the vector fields on $\bar{M}$ and by $C^\infty(\bar{M}, \mathbb{R})$ the algebra of C^∞ real-valued functions on $\bar{M}$. A *Jacobi structure* on $\bar{M}$ is

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a pair $(\bar{\Lambda}, E)$ where $\bar{\Lambda}$ is a 2-vector and E a vector field on $\bar{M}$ verifying

$$[\bar{\Lambda}, \bar{\Lambda}] = 2E \wedge \bar{\Lambda} \qquad [E, \bar{\Lambda}] = 0. \tag{1}$$

Here [,] is the *Schouten–Nijenhuis bracket* (for the definition and properties of the Schouten–Nijenhuis bracket, we refer to [20]). The manifold $\bar{M}$ endowed with a Jacobi structure is called a *Jacobi manifold*. If $(\bar{M}, \bar{\Lambda}, E)$ is a Jacobi manifold we can define a bracket of functions (called *Jacobi bracket*) as follows:

$$\{\bar{f}, \bar{g}\} = \bar{\Lambda}(\mathrm{d}\bar{f}, \mathrm{d}\bar{g}) + \bar{f}E(\bar{g}) - \bar{g}E(\bar{f}) \qquad \text{for all } \bar{f}, \bar{g} \in C^\infty(\bar{M}, \mathbb{R}). \tag{2}$$

The space $C^\infty(\bar{M}, \mathbb{R})$ endowed with the Jacobi bracket is a *local Lie algebra* in the sense of Kirillov (see [10]). Conversely, a structure of local Lie algebra on the space $C^\infty(\bar{M}, \mathbb{R})$ of real-valued functions on a manifold $\bar{M}$ defines a Jacobi structure on $\bar{M}$ (see [10, 8]).

Let $(\bar{M}, \bar{\Lambda}, E)$ be a Jacobi manifold. Define a mapping $\bar{\#} : \Omega^1(\bar{M}) \to \mathfrak{X}(\bar{M})$ from the space of 1-forms on $\bar{M}$, $\Omega^1(\bar{M})$, onto $\mathfrak{X}(\bar{M})$ as follows:

$$(\bar{\#}\bar{\alpha})(\bar{\beta}) = \bar{\Lambda}(\bar{\alpha}, \bar{\beta}) \tag{3}$$

for $\bar{\alpha}, \bar{\beta} \in \Omega^1(\bar{M})$. The mapping $\bar{\#}$ can be extended to a mapping, which we also denote by $\bar{\#}$, from the space of p-forms $\Omega^p(\bar{M})$ on $\bar{M}$ onto the space of p-vectors $\mathcal{V}^p(\bar{M})$ by putting

$$\bar{\#}(\bar{f}) = \bar{f} \qquad \bar{\#}(\bar{\alpha})(\bar{\alpha}_1, \ldots, \bar{\alpha}_p) = (-1)^p \bar{\alpha}(\bar{\#}\bar{\alpha}_1, \ldots, \bar{\#}\bar{\alpha}_p) \tag{4}$$

for $\bar{f} \in C^\infty(\bar{M}, \mathbb{R})$, $\bar{\alpha} \in \Omega^p(\bar{M})$ and $\bar{\alpha}_1, \ldots, \bar{\alpha}_p \in \Omega^1(\bar{M})$.

If $\bar{f} \in C^\infty(\bar{M}, \mathbb{R})$, the vector field $X_{\bar{f}}$ defined by $X_{\bar{f}} = \bar{\#}(\mathrm{d}\bar{f}) + \bar{f}E$ is called the *Hamiltonian vector field* associated with $\bar{f}$. It should be noted that the Hamiltonian vector field associated with the constant function 1 is just E. A direct computation shows that $[X_{\bar{f}}, X_{\bar{g}}] = X_{\{\bar{f},\bar{g}\}}$ (see [14, 15]).

If the vector field E vanishes, { , } is a derivation in each argument, i.e. { , } defines a *Poisson bracket* on $\bar{M}$. In this case, (1) reduces to $[\bar{\Lambda}, \bar{\Lambda}] = 0$, and $(\bar{M}, \bar{\Lambda})$ is a *Poisson manifold*. Poisson and Jacobi manifolds were introduced by Lichnerowicz (see [12–14]).

Now, let (M, Λ) be a Poisson manifold. We can define the contravariant exterior derivative $\sigma : \mathcal{V}^p(M) \to \mathcal{V}^{p+1}(M)$ by $\sigma(P) = -[\Lambda, P]$. Since $\sigma^2 = 0$, σ defines a cohomology on M which is called the *Lichnerowicz–Poisson* (LP for simplicity) *cohomology* for the Poisson manifold M (see [12]). The pth LP-cohomology group is then given by

$$H^p_{\mathrm{LP}}(M) = \frac{\ker\{\sigma : \mathcal{V}^p(M) \to \mathcal{V}^{p+1}(M)\}}{\mathrm{Im}\{\sigma : \mathcal{V}^{p-1}(M) \to \mathcal{V}^p(M)\}}.$$

Notice that $\sigma(\Lambda) = 0$ and thus Λ defines a cohomology class $[\Lambda] \in H^2_{\mathrm{LP}}(M)$. Moreover, if $\# : \Omega^p(M) \to \mathcal{V}^p(M)$ is the mapping given by (4) then, using that $\sigma \circ \# = -\# \circ d$ (see [12, 20]), we have induced homomorphisms in cohomology $\# : H^p_{\mathrm{dR}}(M) \to H^p_{\mathrm{LP}}(M)$, $H^p_{\mathrm{dR}}(M)$ being the pth de Rham cohomology group of M.

It is well known that there is a one-to-one correspondence between the equivalence classes of principal circle bundles over a manifold M and the cohomology group $H^2(M, \mathbb{Z})$. In fact, if Ω is an integer closed 2-form on M then there exists a principal circle bundle $\pi : \bar{M} \to M$ over M with connection form θ such that Ω is the curvature of the connection θ, that is, $\pi^*\Omega = \mathrm{d}\theta$ (see [11]).

A Poisson manifold (M, Λ) admits a *prequantization bundle* (see [20]) if there is a principal circle bundle $\pi : \bar{M} \to M$ over M corresponding to an integer closed 2-form Ω on M such that $\#[\Omega] = [\Lambda]$. Notice that the above condition can also be expressed as follows: (M, Λ) has a prequantization bundle if and only if there exists a vector field A on M, and an integer closed 2-form Ω on M, such that

$$\Lambda + \mathcal{L}_A \Lambda = \#\Omega$$

where $\mathcal{L}$ is the Lie derivative on M.

2.2. Examples of Jacobi manifolds

Next, we will present some examples of Jacobi manifolds which are Poisson manifolds.

2.2.1. Lie–Poisson structures. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with Lie bracket $[,]$ and denote by $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$. Given two functions $f, g \in C^\infty(\mathfrak{g}^*, \mathbb{R})$, we define $\{f, g\}$ as follows. For a point $x \in \mathfrak{g}^*$, we linearize f and g, namely, we take the tangent maps $\mathrm{d}f(x)$ and $\mathrm{d}g(x)$ at x and identify them with two elements $\hat{f}, \hat{g} \in \mathfrak{g}$. Thus, $[\hat{f}, \hat{g}] \in \mathfrak{g}$, and we define

$$\{f, g\}(x) = \langle[\hat{f}, \hat{g}], x\rangle.$$

$\{,\}$ is the so-called *Lie–Poisson bracket* on $\mathfrak{g}^*$. If x^i are global coordinates for $\mathfrak{g}^*$ obtained from a basis, we have

$$\Lambda = \sum_{i<j} \Lambda^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \tag{5}$$

where

$$\Lambda^{ij} = \sum_k C_k^{ij} x^k \tag{6}$$

C_k^{ij} being the structure constants of $\mathfrak{g}$.

2.2.2. Symplectic manifolds. A *symplectic manifold* is a pair (M, Ω), where M is an even-dimensional manifold and Ω is a closed non-degenerate 2-form on M. We define a Poisson 2-vector Λ on M by

$$\Lambda(\alpha, \beta) = \Omega(\flat^{-1}(\alpha), \flat^{-1}(\beta)) \tag{7}$$

for all $\alpha, \beta \in \Omega^1(M)$, where $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ is the isomorphism of $C^\infty(M, \mathbb{R})$-modules given by $\flat(X) = i_X\Omega$. Notice that $\flat^{-1} = -\#$ and that $\Lambda = \#\Omega$. Moreover, in this case, the homomorphism $\# : H^p_{\mathrm{dR}}(M) \to H^p_{\mathrm{LP}}(M)$ (see section 2.1) is an isomorphism (see [7]). If we choose canonical coordinates (q^i, p_i) on M, we have

$$\Omega = \sum_i \mathrm{d}q^i \wedge \mathrm{d}p_i \qquad \Lambda = \sum_i \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}.$$

2.2.3. Cosymplectic manifolds. A *cosymplectic manifold* (see [1, 2, 4]) is a triple (M, Ω, η), where M is an odd-dimensional manifold, Ω is a closed 2-form and η is a closed 1-form on M such that $\eta \wedge \Omega^m$ is a volume form, with $\dim M = 2m + 1$. If $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ is the isomorphism of $C^\infty(M, \mathbb{R})$-modules defined by $\flat(X) = i_X\Omega + (i_X\eta)\eta$, then the vector field $\xi = \flat^{-1}(\eta)$ is called the *Reeb vector field* of M. The vector field ξ is characterized by the relations $i_\xi\Omega = 0$ and $i_\xi\eta = 1$. In particular, $\mathcal{L}_\xi\Omega = 0$ and $\mathcal{L}_\xi\eta = 0$. A 2-vector Λ on M is defined by

$$\Lambda(\alpha, \beta) = \Omega(\flat^{-1}(\alpha), \flat^{-1}(\beta)) = \Omega(\flat^{-1}(\alpha - \alpha(\xi)\eta), \flat^{-1}(\beta - \beta(\xi)\eta)) \tag{8}$$

for all $\alpha, \beta \in \Omega^1(M)$. Thus, (M, Λ) becomes a Poisson manifold. It should be noticed that $\#\alpha = -\flat^{-1}(\alpha) + \alpha(\xi)\xi$, for $\alpha \in \Omega^1(M)$, and that $\#\Omega = \Lambda$. Moreover, $\mathcal{L}_\xi \Lambda = 0$. In canonical coordinates $\{q^1, \ldots, q^m, p_1, \ldots, p_m, z\}$, we have ([1, 4])

$$\Omega = \sum_{i=1}^{m} \mathrm{d}q^i \wedge \mathrm{d}p_i \qquad \eta = \mathrm{d}z \qquad \xi = \frac{\partial}{\partial z} \qquad \Lambda = \sum_{i=1}^{m} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}.$$

Other very interesting examples of Jacobi manifolds, which are not Poisson manifolds, are the contact and the locally conformal symplectic manifolds that we will next describe.

2.2.4. Contact manifolds. Let $\bar{M}$ be a $2m+1$-dimensional manifold and θ a 1-form on $\bar{M}$. We said that θ is a contact 1-form if $\theta \wedge (\mathrm{d}\theta)^m \neq 0$ at every point. In such a case $(\bar{M}, \theta)$ is termed a *contact manifold* (see, for instance, [1, 2]). Using the classical theorem of Darboux, around every point of $\bar{M}$ there exists canonical coordinates $(t, q^1, \ldots, q^m, p_1, \ldots, p_m)$ such that

$$\theta = \mathrm{d}t - \sum_i p_i \, \mathrm{d}q^i.$$

A contact manifold $(\bar{M}, \theta)$ is a Jacobi manifold. In fact, we define the 2-vector $\bar{\Lambda}$ by

$$\bar{\Lambda}(\bar{\alpha}, \bar{\beta}) = \mathrm{d}\theta(\bar{\flat}^{-1}(\bar{\alpha}), \bar{\flat}^{-1}(\bar{\beta})) \tag{9}$$

for all $\bar{\alpha}, \bar{\beta} \in \Omega^1(\bar{M})$, where $\bar{\flat} : \mathfrak{X}(\bar{M}) \to \Omega^1(\bar{M})$ is the isomorphism of $C^\infty(\bar{M}, \mathbb{R})$-modules given by $\bar{\flat}(\bar{X}) = i_{\bar{X}} \, \mathrm{d}\theta + \theta(\bar{X})\theta$. The vector field E is just the Reeb vector field $\xi = \bar{\flat}^{-1}(\theta)$ of $(\bar{M}, \theta)$. The vector field ξ is characterized by the relations $i_\xi \theta = 1$ and $i_\xi \, \mathrm{d}\theta = 0$. A direct computation shows that $\bar{\#}\bar{\alpha} = -\bar{\flat}^{-1}\bar{\alpha} + \bar{\alpha}(\xi)\xi$, for $\bar{\alpha} \in \Omega^1(\bar{M})$ and that $\bar{\#}(\mathrm{d}\theta) = \bar{\Lambda}$. Using canonical coordinates we get

$$\bar{\Lambda} = \sum_i \left(\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial t} \right) \wedge \frac{\partial}{\partial p_i} \qquad E = \frac{\partial}{\partial t}.$$

2.2.5. Locally conformal symplectic manifolds. An *almost symplectic manifold* is a pair $(\bar{M}, \Phi)$, where $\bar{M}$ is an even-dimensional manifold and Φ is a non-degenerate 2-form on $\bar{M}$. An almost symplectic manifold is said to be *locally conformal symplectic (LCS)* if for each point $\bar{x} \in \bar{M}$ there is an open neighbourhood $\bar{U}$ such that $\mathrm{d}(e^{-\sigma}\Phi) = 0$, for some function $\sigma : \bar{U} \to \mathbb{R}$ (see for example [19]). Equivalently, $(\bar{M}, \Phi)$ is a LCS manifold if there exists a closed 1-form ω such that

$$\mathrm{d}\Phi = \omega \wedge \Phi. \tag{10}$$

The 1-form ω is called the *Lee 1-form* of $\bar{M}$. It is obvious that the LCS manifolds with Lee 1-form identically zero are just the symplectic manifolds.

In a similar way that for contact manifolds, we define a 2-vector $\bar{\Lambda}$ and a vector field E on $\bar{M}$, which are given by

$$\bar{\Lambda}(\bar{\alpha}, \bar{\beta}) = \Phi(\bar{\flat}^{-1}(\bar{\alpha}), \bar{\flat}^{-1}(\bar{\beta})) \qquad E = \bar{\flat}^{-1}\omega \tag{11}$$

for all 1-forms $\bar{\alpha}$ and $\bar{\beta}$, where $\bar{\flat} : \mathfrak{X}(\bar{M}) \to \Omega^1(\bar{M})$ is the isomorphism of $C^\infty(\bar{M}, \mathbb{R})$-modules defined by $\bar{\flat}(\bar{X}) = i_{\bar{X}}\Phi$. Then $(\bar{M}, \bar{\Lambda}, E)$ is a Jacobi manifold (see [6, 10, 14]). Now, $\bar{\#} = -\bar{\flat}^{-1}$ and $\bar{\#}\Phi = \bar{\Lambda}$. Notice that

$$\omega(E) = 0 \qquad \mathcal{L}_E \omega = 0 \qquad \mathcal{L}_E \Phi = 0. \tag{12}$$

2.3. The characteristic foliation of a Jacobi manifold

We will show that an arbitrary Jacobi manifold is foliated by leaves which are contact or LCS manifolds. Roughly speaking, a Jacobi manifold is made of contact or LCS pieces.

A Jacobi manifold $(\bar{M}, \bar{\Lambda}, E)$ is said to be *transitive* if for all $\bar{x} \in \bar{M}$ the tangent space $T_{\bar{x}}\bar{M}$ is generated by $\bar{\#}_{\bar{x}}(T^*_{\bar{x}}\bar{M})$ and $E_{\bar{x}}$ [6].

Let $(\bar{M}, \bar{\Lambda}, E)$ be a transitive Jacobi manifold. Then we have the following (see [6] and the references therein).

(a) If $\dim \bar{M} = 2m+1$, it follows that $T_{\bar{x}}\bar{M} = \bar{\#}_{\bar{x}}(T^*_{\bar{x}}\bar{M}) \oplus \langle E_{\bar{x}} \rangle$ for all $\bar{x} \in \bar{M}$. Therefore, the 1-form θ defined by $\theta_{\bar{x}}(\bar{u} + \lambda E_{\bar{x}}) = \lambda$, for $\bar{u} \in \bar{\#}_{\bar{x}}(T^*_{\bar{x}}\bar{M})$ and $\lambda \in \mathbb{R}$, is a contact 1-form.

(b) If $\dim \bar{M} = 2m$, we deduce that $\bar{\#}_{\bar{x}} : T^*_{\bar{x}}\bar{M} \to T_{\bar{x}}\bar{M}$ is an isomorphism. Thus, if we put

$$\Phi_{\bar{x}}(\bar{X}, \bar{Y}) = \bar{\Lambda}_{\bar{x}}((\bar{\#}_{\bar{x}})^{-1}\bar{X}, (\bar{\#}_{\bar{x}})^{-1}\bar{Y}) \qquad \text{for all } \bar{X}, \bar{Y} \in T_{\bar{x}}\bar{M}$$

and $\omega_{\bar{x}} = i_{E_{\bar{x}}}\Phi_{\bar{x}}$, we get that $(\bar{M}, \Phi)$ is a LCS manifold with Lee 1-form ω.

Consequently, the transitive Jacobi manifolds are just the contact or LCS manifolds. It should be noticed that the transitive Poisson manifolds are just the symplectic manifolds.

Now, let $(\bar{M}, \bar{\Lambda}, E)$ be an arbitrary Jacobi manifold. Denote by $\bar{D}_{\bar{x}}$ the subspace of $T_{\bar{x}}\bar{M}$ generated by all the Hamiltonian vector fields evaluated at the point $\bar{x} \in \bar{M}$. In other words, $\bar{D}_{\bar{x}} = \bar{\#}_{\bar{x}}(T^*_{\bar{x}}\bar{M}) + \langle E_{\bar{x}} \rangle$. Since $\bar{D}$ is involutive, one easily follows that $\bar{D}$ defines a generalized foliation in the sense of Sussmann [17]. This foliation is termed the *characteristic foliation* in [6]. Moreover, if $\bar{L}$ is a leaf of $\bar{D}$, the Jacobi structure $(\bar{\Lambda}, E)$ on $\bar{M}$ induces a transitive Jacobi structure $(\bar{\Lambda}_{\bar{L}}, E_{\bar{L}})$ on $\bar{L}$. Thus, we deduce that the leaves of $\bar{D}$ are contact or LCS manifolds (for a detailed study we refer to [6]). If $\bar{M}$ is a Poisson manifold then the leaves of $\bar{D}$ are symplectic manifolds. In particular, for a cosymplectic manifold (M, Ω, η) the characteristic foliation is the foliation of codimension 1 given by $\eta = 0$ (see [1, 4]). For a Lie–Poisson structure the leaves of the symplectic foliation are the coadjoint orbits (see, for instance, [20]).

2.4. Regular Jacobi manifolds

A Jacobi manifold $(\bar{M}, \bar{\Lambda}, E)$ is said to be *regular* ([5]) if the vector field E is complete, $E \neq 0$ at every point and the one-dimensional foliation defined by E is regular in the sense of Palais [16]. In such a case, the space of leaves $M = \bar{M}/E$ has a structure of differentiable manifold, and the canonical projection $\pi : \bar{M} \to M$ is a fibration (surjective submersion). Moreover, we can define on M a 2-vector Λ by $\Lambda(\alpha, \beta) \circ \pi = \bar{\Lambda}(\pi^*\alpha, \pi^*\beta)$, for all $\alpha, \beta \in \Omega^1(M)$. Notice that, from (1), Λ is well defined and (M, Λ) is a Poisson manifold (see [6]). The relationship between the corresponding Jacobi and Poisson brackets $\{,\}_{\bar{M}}$ and $\{,\}_M$ is given by the following formula:

$$\{\pi^*f, \pi^*g\}_{\bar{M}} = \{f, g\}_M \circ \pi \qquad \text{for all } f, g \in C^\infty(M, \mathbb{R}).$$

Thus, if f is a function on M, the Hamiltonian vector fields defined by f and π^*f are π-related. In particular, if $\bar{D}$ and D are the characteristic foliations of $\bar{M}$ and M, respectively, then $D_{\pi(\bar{x})} \subseteq \pi_*^{\bar{x}}(\bar{D}_{\bar{x}}) = \pi_*^{\bar{x}}(\bar{\#}_{\bar{x}}(T^*_{\bar{x}}\bar{M}))$, for $\bar{x} \in \bar{M}$.

3. Prequantizable Poisson manifolds and Jacobi structures

Let $\pi : \bar{M} \to M$ be a principal circle bundle over a manifold M endowed with a connection form θ.

Suppose that P is a p-vector on M. We define the horizontal lift of P to $\bar{M}$ as the p-vector P^{H} on $\bar{M}$ characterized by the following conditions:

$$P^{\mathrm{H}}(\pi^*\alpha_1, \ldots, \pi^*\alpha_p) = P(\alpha_1, \ldots, \alpha_p) \circ \pi \qquad i_\theta P^{\mathrm{H}} = 0 \tag{13}$$

for $\alpha_1, \ldots, \alpha_p \in \Omega^1(M)$. Notice that if $P = X_1 \wedge \ldots \wedge X_p$ with $X_i \in \mathfrak{X}(M)$ then

$$P^{\mathrm{H}} = X_1^{\mathrm{H}} \wedge \ldots \wedge X_p^{\mathrm{H}} \tag{14}$$

where X_i^{H}, $1 \leqslant i \leqslant p$, is the horizontal lift of the vector field X_i to $\bar{M}$.

Theorem 3.1. Let (M, Λ) be a Poisson manifold which admits a prequantization bundle $\pi : \bar{M} \to M$. Then, there exists on $\bar{M}$ a Jacobi structure $(\bar{\Lambda}, E)$ and a 1-form θ such that:

(i) $(\bar{M}, \bar{\Lambda}, E)$ is a regular Jacobi manifold and the corresponding quotient Poisson manifold $\bar{M}/E$ is just M;

(ii) the vector field E satisfies

$$\theta(E) = 1 \qquad \mathcal{L}_E\theta = 0. \tag{15}$$

Proof. Let θ be a connection form in the bundle $\pi : \bar{M} \to M$ with curvature form $\Omega \in \Omega^2(M)$ ($[\Omega] \in H^2(M, \mathbb{Z})$) and let A be a vector field on M such that

$$\Lambda + \mathcal{L}_A\Lambda = \#\Omega. \tag{16}$$

Denote by E the infinitesimal generator of the action of S^1 on $\bar{M}$. Then, it is clear that $\theta(E) = 1$ and $\mathcal{L}_E\theta = 0$.

Now, we consider on $\bar{M}$ the 2-vector $\bar{\Lambda}$ defined by

$$\bar{\Lambda} = \Lambda^{\mathrm{H}} + E \wedge A^{\mathrm{H}}. \tag{17}$$

From (13) and (15), we deduce that

$$\mathcal{L}_E\Lambda^{\mathrm{H}} = 0. \tag{18}$$

Therefore, since

$$[E, A^{\mathrm{H}}] = 0 \tag{19}$$

we obtain that $\mathcal{L}_E\bar{\Lambda} = [E, \bar{\Lambda}] = 0$.

Next, we will prove that $[\bar{\Lambda}, \bar{\Lambda}] = 2E \wedge \bar{\Lambda}$. If X, Y are vector fields on M we have that

$$[X^{\mathrm{H}}, Y^{\mathrm{H}}] = [X, Y]^{\mathrm{H}} - (\Omega(X, Y) \circ \pi)E. \tag{20}$$

Thus, using (3), (4), (14), (19) and (20), we conclude that

$$\begin{aligned} &[\Lambda^{\mathrm{H}}, \Lambda^{\mathrm{H}}] = [\Lambda, \Lambda]^{\mathrm{H}} + 2E \wedge (\#\Omega)^{\mathrm{H}} = 2E \wedge (\#\Omega)^{\mathrm{H}} \\ &E \wedge A^{\mathrm{H}}, E \wedge A^{\mathrm{H}}] = 0. \end{aligned} \tag{21}$$

On the other hand, from (18), we deduce that

$$[\Lambda^{\mathrm{H}}, E \wedge A^{\mathrm{H}}] = -E \wedge \mathcal{L}_{A^{\mathrm{H}}}\Lambda^{\mathrm{H}}. \tag{22}$$

Moreover, a direct computation, using (13), (16), (19) and (20) shows that

$$(\mathcal{L}_{A^{\mathrm{H}}}\Lambda^{\mathrm{H}})(\pi^*\alpha, \pi^*\beta) = (-\Lambda^{\mathrm{H}} + (\#\Omega)^{\mathrm{H}})(\pi^*\alpha, \pi^*\beta)$$

for $\alpha, \beta \in \Omega^1(M)$ which, because (22), implies that

$$[\Lambda^{\mathrm{H}}, E \wedge A^{\mathrm{H}}] = E \wedge \Lambda^{\mathrm{H}} - E \wedge (\#\Omega)^{\mathrm{H}}. \tag{23}$$

Consequently, from (17), (21) and (23), we obtain that $[\bar{\Lambda}, \bar{\Lambda}] = 2E \wedge \bar{\Lambda}$.

Finally, it is obvious that $\bar{\Lambda}(\pi^*\alpha, \pi^*\beta) = \Lambda(\alpha, \beta) \circ \pi$, for $\alpha, \beta \in \Omega^1(M)$ (see (13) and (17)). □

Next, we will prove a converse of theorem 3.1.

Theorem 3.2. Let $(\bar{M}, \bar{\Lambda}, E)$ be a compact regular Jacobi manifold and θ a 1-form on $\bar{M}$ such that $\theta(E) = 1$ and $\mathcal{L}_E\theta = 0$. If $M = \bar{M}/E$ is the induced quotient Poisson manifold then M admits a prequantization bundle $\pi : \bar{M} \to M$, π being the canonical projection, and the induced Jacobi structure on $\bar{M}$ is just $(\bar{\Lambda}, E)$.

Proof. Using the results of [18], we deduce that the period function of E is constant. Assume, for the sake of simplicity, that the period function of E is equal to 1. Then $\bar{M}$ is a principal circle bundle over M, E is the infinitesimal generator of the action of S^1 on $\bar{M}$ and θ is a connection form in the bundle (see [18]). Suppose that $\Omega \in \Omega^2(M)$ is the curvature form of the connection θ, that is,

$$\mathrm{d}\theta = \pi^*\Omega. \tag{24}$$

Now, we consider on $\bar{M}$ the vector field $\bar{A}$ given by

$$\bar{A} = i_\theta\bar{\Lambda}. \tag{25}$$

From (1) and (25), we deduce that $\theta(\bar{A}) = 0$ and $[E, \bar{A}] = i_{(\mathcal{L}_E\theta)}\bar{\Lambda} + i_\theta\mathcal{L}_E\bar{\Lambda} = 0$, which implies that there exists a vector field A on M such that

$$\bar{A} = A^{\mathrm{H}}. \tag{26}$$

A direct computation shows that (see (20))

$$(\mathcal{L}_A\Lambda)(\alpha, \beta) \circ \pi = (\mathcal{L}_{A^{\mathrm{H}}}\bar{\Lambda})(\pi^*\alpha, \pi^*\beta) \tag{27}$$

for $\alpha, \beta \in \Omega^1(M)$, where Λ is the Poisson 2-vector of M.

Using (1), (25) and (26), we have that

$$\mathcal{L}_{A^{\mathrm{H}}}\bar{\Lambda} = -\tfrac{1}{2}i_\theta[\bar{\Lambda}, \bar{\Lambda}] + (\bar{\#}(\mathrm{d}\theta)) = -\bar{\Lambda} + E \wedge A^{\mathrm{H}} + (\bar{\#}(\mathrm{d}\theta)). \tag{28}$$

On the other hand, if $\alpha \in \Omega^1(M)$, since the vector field $\bar{\#}\pi^*\alpha$ is π-projectable onto the vector field $\#\alpha$ (see section 2.4), from (4), (24) and (28), we deduce that

$$(\mathcal{L}_{A^{\mathrm{H}}}\bar{\Lambda})(\pi^*\alpha, \pi^*\beta) = (-\Lambda(\alpha, \beta) + (\#\Omega)(\alpha, \beta)) \circ \pi. \tag{29}$$

Consequently, using (27) and (29), we conclude that

$$\Lambda + \mathcal{L}_A\Lambda = \#\Omega.$$

Finally, from (13), (25) and (26), we obtain that

$$\Lambda^{\mathrm{H}} = \bar{\Lambda} - E \wedge A^{\mathrm{H}}$$

that is, $\bar{\Lambda} = \Lambda^{\mathrm{H}} + E \wedge A^{\mathrm{H}}$. This ends the proof of our result. □

Let (M, Λ) be a prequantizable Poisson manifold and $\pi : \bar{M} \to M$ a prequantization bundle. Using theorem 3.1, (17), (25), (26) and the fact that the Hamiltonian vector field X_{π^*f} on $\bar{M}$ is π-projectable onto the Hamiltonian vector field X_f on M, for $f \in C^\infty(M, \mathbb{R})$, we have

Corollary 3.3. Let (M, Λ) be a prequantizable Poisson manifold and $\pi : \bar{M} \to M$ a prequantization bundle. Suppose that A is a vector field on M which satisfies (16) and that $(\bar{\Lambda}, E)$ is the associated Jacobi structure on $\bar{M}$.

(i) If $f, g \in C^\infty(M, \mathbb{R})$ then

$$\{\pi^* f, \pi^* g\}_{\bar{M}} = \{f, g\}_M \circ \pi$$

where $\{,\}_{\bar{M}}$ (respectively $\{,\}_M$) is the Jacobi bracket (respectively Poisson bracket) on $\bar{M}$ (respectively M).

(ii) If X_f (respectively $X_{\pi^* f}$) is the Hamiltonian vector field on M (respectively $\bar{M}$) associated with the function f (respectively $\pi^* f$) then

$$X_{\pi^* f} = X_f^{\mathrm{H}} + (\pi^* f - A(f) \circ \pi)E.$$

(iii) If $\bar{D}$ (respectively D) is the characteristic foliation of $\bar{M}$ (respectively M) then

$$\bar{D} = (D^{\mathrm{H}} + \langle A^{\mathrm{H}} \rangle) \oplus \langle E \rangle$$

where D^{H} is the horizontal lift to $\bar{M}$ of the foliation D.

4. Examples

Example 4.1. *(Lie–Poisson structures.)* The Lie–Poisson structure Λ on the dual space $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ is prequantizable, since it is exact. In fact, if x^i are global coordinates for $\mathfrak{g}^*$ obtained from a basis, we have that $\Lambda = \sigma(A)$, where

$$A = \sum_i x^i \frac{\partial}{\partial x^i}.$$

Thus, the prequantization bundle is trivial, say $\bar{M} = \mathfrak{g}^* \times S^1$, the connection is flat and, if E is the canonical vector field on S^1, then the induced Jacobi structure $(\bar{\Lambda}, E)$ on $\bar{M}$ is given by

$$\bar{\Lambda} = \sum_{\substack{i,j,k \\ i<j}} C_k^{ij} x^k \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + \sum_i x^i E \wedge \frac{\partial}{\partial x^i}.$$

So, $\mathfrak{g}^* \times S^1$ is endowed with a linear Jacobi structure.

Example 4.2. *(Symplectic manifolds.)* Let (M, Ω) be a symplectic manifold and Λ the Poisson 2-vector on M defined by (7). Using the results of section 2.2, we deduce that M is prequantizable as a Poisson manifold if and only if M is prequantizable as a symplectic manifold. But the necessary and sufficient condition to (M, Ω) be prequantizable is that $[\Omega]$ would define an element of the integer cohomology group $H^2(M, \mathbb{Z})$.

Suppose that (M, Ω) is prequantizable and that $\pi : \bar{M} \to M$ is a prequantization bundle. Then, the pair $(\bar{\Lambda}, E)$ is a Jacobi structure on $\bar{M}$, where $\bar{\Lambda} = \Lambda^{\mathrm{H}}$ and E is the infinitesimal generator of the action of S^1 on $\bar{M}$ (notice that, in this case, a vector field on M satisfying (16) is just the vector field $A = 0$).

Now, let θ be a connection form in the principal circle bundle $\pi : \bar{M} \to M$ such that $\pi^*\Omega = \mathrm{d}\theta$. A direct computation shows that θ is a contact 1-form on $\bar{M}$. The Reeb vector field of the contact structure is E. Denote by $\bar{\flat} : \mathfrak{X}(\bar{M}) \to \Omega^1(\bar{M})$ the isomorphism of $C^\infty(\bar{M}, \mathbb{R})$-modules given by $\bar{\flat}(\bar{X}) = i_{\bar{X}} \mathrm{d}\theta + \theta(\bar{X})\theta$ and by $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ the isomorphism of $C^\infty(M, \mathbb{R})$-modules defined by $\flat(X) = i_X \Omega$. Then, we have that $\bar{\flat} X^{\mathrm{H}} = \pi^*(\flat X)$, for $X \in \mathfrak{X}(M)$, which implies that

$$\bar{\flat}^{-1} \pi^* \alpha = (\flat^{-1} \alpha)^{\mathrm{H}} \tag{30}$$

for $\alpha \in \Omega^1(M)$. Thus, if $(\tilde{\Lambda}, E)$ is the Jacobi structure defined by the contact 1-form θ on $\bar{M}$ we obtain, using (7), (9), (13) and (30), that

$$\tilde{\Lambda}(\pi^*\alpha, \pi^*\beta) = \mathrm{d}\theta((\flat^{-1}\alpha)^{\mathrm{H}}, (\flat^{-1}\beta)^{\mathrm{H}}) = \Lambda(\alpha, \beta) \circ \pi = \Lambda^{\mathrm{H}}(\pi^*\alpha, \pi^*\beta)$$

for $\alpha, \beta \in \Omega^1(M)$. Therefore, since $i_\theta \tilde{\Lambda} = 0$, we deduce that $\bar{\Lambda} = \tilde{\Lambda}$ (see (13) and (9)).

Finally, if X_f (respectively $X_{\pi^* f}$) is the Hamiltonian vector field on M (respectively $\bar{M}$) associated with the function f (respectively $\pi^* f$) then, from corollary 3.3, we conclude that

$$X_{\pi^* f} = X_f^{\mathrm{H}} + (\pi^* f) E.$$

Next, we will study two particular cases: the two-dimensional unit sphere S^2 and the two-dimensional real torus $\mathbb{T}^2$.

(i) *The two-dimensional unit sphere S^2 (see [9]).* Let $i : S^2 \to \mathbb{R}^3$ be the canonical inclusion and (x, y, z) the usual coordinates in $\mathbb{R}^3$. We consider on S^2 the symplectic 2-form Ω defined by

$$\Omega = \frac{1}{4\pi} i^*(x \mathrm{d}y \wedge \mathrm{d}z - y\, \mathrm{d}x \wedge \mathrm{d}z + z\, \mathrm{d}x \wedge \mathrm{d}y).$$

$[\Omega]$ is a generator of the integer cohomology group $H^2(S^2, \mathbb{Z}) = \mathbb{Z}$. Thus, the symplectic manifold (S^2, Ω) is prequantizable. The corresponding Poisson bracket on S^2 is given by

$$\{i^*x, i^*y\} = (4\pi) i^*z \qquad \{i^*x, i^*z\} = -(4\pi) i^*y \qquad \{i^*y, i^*z\} = (4\pi) i^*x.$$

Denote by S^3 the three-dimensional unit sphere in $\mathbb{R}^4$

$$S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 / (x_1)^2 + (x_2)^2 + (x_3)^2 + (x_4)^2 = 1\}.$$

It is well known that S^3 is the total space of the principal circle bundle over S^2 corresponding to the integer closed 2-form Ω. In fact, the projection of the bundle is the *Hopf fibration* $\pi : S^3 \to S^2$ and the action of S^1 on S^3 is the usual. Notice that (see, for instance, [3])

$$\pi^*\Omega = \frac{1}{\pi} j^*(\mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_3 \wedge \mathrm{d}x_4) \tag{31}$$

where $j : S^3 \to \mathbb{R}^4$ is the canonical inclusion.

Now, using the diffeomorphism between S^3 and the special unitary group $SU(2)$ given by

$$S^3 \to SU(2) \qquad (x_1, x_2, x_3, x_4) \to \begin{pmatrix} x_1 + \mathrm{i}x_2 & -x_3 + \mathrm{i}x_4 \\ x_3 + \mathrm{i}x_4 & x_1 - \mathrm{i}x_2 \end{pmatrix}$$

we will describe the contact and Jacobi structures induced on $SU(2)$.

Denote by $\mathfrak{su}(2) = \{A \in gl(2, \mathbb{C}) / \bar{A}^T = -A, \text{trace } X = 0\}$ the Lie algebra of $SU(2)$ and by σ_1, σ_2 and σ_3 the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then, the matrices $\{\sqrt{\pi}\mathrm{i}\sigma_1, \sqrt{\pi}\mathrm{i}\sigma_2, 2\pi\mathrm{i}\sigma_3\}$ form a basis of $\mathfrak{su}(2)$ which defines on $SU(2)$ a basis of left invariant vector fields $\{X, Y, \xi\}$. Suppose that $\{\alpha, \beta, \theta\}$ is the dual basis of left invariant 1-forms. A direct computation shows that

$$[X, Y] = -\xi \qquad [X, \xi] = 4\pi Y \qquad [Y, \xi] = -4\pi X. \tag{32}$$

Moreover, it is clear that ξ is the infinitesimal generator of the action of S^1 on $SU(2)$. Also, from (32), we have that

$$\mathrm{d}\alpha = 4\pi\beta \wedge \theta \qquad \mathrm{d}\beta = -4\pi\alpha \wedge \theta \qquad \mathrm{d}\theta = \alpha \wedge \beta.$$

In particular, $\mathcal{L}_\xi\theta = 0$. Thus, θ is a connection form in the principal circle bundle $\pi : SU(2) \simeq S^3 \to S^2$ and, since the 2-form $\pi^*\Omega$ is left invariant (see (31)), we deduce that

$$\mathrm{d}\theta = \alpha \wedge \beta = \pi^*\Omega. \tag{33}$$

Consequently, the Jacobi structure $(\bar{\Lambda}, E)$ defined by the contact 1-form θ on $SU(2)$ is just the Jacobi structure on $SU(2)$ induced by the prequantizable symplectic manifold (S^2, Ω). In fact, using (33), we conclude that

$$\bar{\Lambda} = X \wedge Y \qquad E = \xi.$$

(ii) *The two-dimensional real torus* $\mathbb{T}^2$. Let $\bar{\Omega}$ be the usual symplectic 2-form on $\mathbb{R}^2$, $\bar{\Omega} = \mathrm{d}q \wedge \mathrm{d}p$, (q, p) being the canonical coordinates on $\mathbb{R}^2$. Denote by Ω the symplectic 2-form on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ induced by $\bar{\Omega}$. $[\Omega]$ is a generator of the integer cohomology group $H^2(\mathbb{T}^2, \mathbb{Z}) = \mathbb{Z}$. Thus, the symplectic manifold $(\mathbb{T}^2, \Omega)$ is prequantizable.

Now, let H be the *Heisenberg group*. It is well known that H is the Lie group of matrices of real numbers of the form

$$A = \begin{pmatrix} 1 & q & t \\ 0 & 1 & p \\ 0 & 0 & 1 \end{pmatrix}$$

with $q, p, t \in \mathbb{R}$. H is a connected simply connected nilpotent Lie group of dimension three. A global system of coordinates (q, p, t) on H is defined by

$$q(A) = q \qquad p(A) = p \qquad t(A) = t.$$

A basis for the right invariant 1-forms on H is given by

$$\tilde{\alpha} = \mathrm{d}q \qquad \tilde{\beta} = \mathrm{d}p \qquad \tilde{\theta} = \mathrm{d}t - p\,\mathrm{d}q \tag{34}$$

and its dual basis of right invariant vector fields on H is given by

$$\tilde{X} = \frac{\partial}{\partial q} + p\frac{\partial}{\partial t} \qquad \tilde{Y} = \frac{\partial}{\partial p} \qquad \tilde{\xi} = \frac{\partial}{\partial t}. \tag{35}$$

Denote by Γ the subgroup of matrices of H with integer entries and by $\bar{M} = H/\Gamma$ the space of left cosets. Then $\bar{M}$ is a compact nilmanifold. The 1-forms $\tilde{\alpha}, \tilde{\beta}, \tilde{\theta}$ and the vector fields $\tilde{X}, \tilde{Y}, \tilde{\xi}$ on H all descend to $\bar{M}$; denote the 1-forms and the vector fields induced on $\bar{M}$ by $\alpha, \beta, \theta, X, Y$ and ξ, respectively.

The space $\bar{M}$ is a principal circle bundle over $\mathbb{T}^2$. The projection π of the bundle is

$$\pi[(q, p, t)] = [(q, p)] \tag{36}$$

and the infinitesimal generator of the action of S^1 on $\bar{M}$ is the vector field ξ. Thus, from (34), (35) and (36), we deduce that θ is a connection form in the principal circle bundle $\pi : \bar{M} \to \mathbb{T}^2$ and

$$\mathrm{d}\theta = \alpha \wedge \beta = \pi^*\Omega.$$

Therefore, the Jacobi structure $(\bar{\Lambda}, E)$ defined on $\bar{M}$ by the contact 1-form θ is just the Jacobi structure induced on $\bar{M}$ by the prequantizable symplectic manifold $(\mathbb{T}^2, \Omega)$. In fact, we have that

$$\bar{\Lambda} = X \wedge Y \qquad E = \xi.$$

Example 4.3. *(Cosymplectic manifolds.)* Let (M, Ω, η) be a cosymplectic manifold with Reeb vector field ξ and Λ the Poisson 2-vector on M defined by (8). Using the results of section 2.2, we have that

$$\Lambda + \mathcal{L}_A \Lambda = \#\Omega \tag{37}$$

with $A = 0$ or $A = \xi$.

Now, assume that $[\Omega]$ defines an element of the integer cohomology group $H^2(M, \mathbb{Z})$. Then, from (37), we deduce that (M, Λ) is a prequantizable Poisson manifold. Suppose that $\pi : \bar{M} \to M$ is a prequantization bundle and let θ be a connection form in the principal circle bundle $\pi : \bar{M} \to M$ such that $\pi^*\Omega = \mathrm{d}\theta$. Denote by E the infinitesimal generator of the action of S^1 on $\bar{M}$. Using theorem 3.1 (see (17)) we obtain that $(\bar{\Lambda}_1, E)$ and $(\bar{\Lambda}_2, E)$ are Jacobi structures on $\bar{M}$, where $\bar{\Lambda}_1$ and $\bar{\Lambda}_2$ are the 2-vectors on $\bar{M}$ given by

$$\bar{\Lambda}_1 = \Lambda^{\mathrm{H}} + E \wedge \xi^{\mathrm{H}} \qquad \bar{\Lambda}_2 = \Lambda^{\mathrm{H}}. \tag{38}$$

Next, we shall study the structures $(\bar{\Lambda}_1, E)$ and $(\bar{\Lambda}_2, E)$.

(i) *The Jacobi structure* $(\bar{\Lambda}_1, E)$. We consider on $\bar{M}$ the 2-form Φ given by

$$\Phi = \mathrm{d}\theta - (\pi^*\eta) \wedge \theta. \tag{39}$$

A direct computation proves that $(\bar{M}, \Phi)$ is a LCS manifold with Lee 1-form $\omega = \pi^*\eta$.

We will show that the Jacobi structure on $\bar{M}$ induced by the LCS 2-form Φ is just $(\bar{\Lambda}_1, E)$. Denote by $\bar{\flat} : \mathfrak{X}(\bar{M}) \to \Omega^1(\bar{M})$ the isomorphism of $C^\infty(\bar{M}, \mathbb{R})$-modules defined by $\bar{\flat}(\bar{X}) = i_{\bar{X}}\Phi$, and by $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ the isomorphism of $C^\infty(M, \mathbb{R})$-modules given by $\flat(X) = i_X\Omega + \eta(X)\eta$. Then, from (39), we have that

$$\pi^*(\flat(X - \eta(X)\xi)) = \bar{\flat}X^{\mathrm{H}} + (\eta(X) \circ \pi)\theta \qquad \bar{\flat}E = \pi^*\eta = \omega \tag{40}$$

for $X \in \mathfrak{X}(M)$, which implies that

$$(\flat^{-1}(\alpha - \alpha(\xi)\eta))^{\mathrm{H}} = \bar{\flat}^{-1}\pi^*\alpha - (\alpha(\xi) \circ \pi)E \tag{41}$$

for $\alpha \in \Omega^1(M)$.

Let $(\tilde{\Lambda}_1, E_1)$ be the Jacobi structure on $\bar{M}$ induced by Φ. It is clear that $E_1 = E$ (see (11) and (40)). Moreover, using (8), (11), (13) and (39)–(41), we deduce that

$$\tilde{\Lambda}_1(\pi^*\alpha, \pi^*\beta) = \Lambda^{\mathrm{H}}(\pi^*\alpha, \pi^*\beta) \qquad i_\theta\tilde{\Lambda}_1 = \xi^{\mathrm{H}} \tag{42}$$

for $\alpha, \beta \in \Omega^1(M)$. Thus, from (38) and (42), we conclude that $\tilde{\Lambda}_1 = \bar{\Lambda}_1$.

Finally, if X_f (respectively X_{π^*f}) is the Hamiltonian vector field on M (respectively $\bar{M}$) associated with the function f (respectively π^*f) then, from corollary 3.3, we obtain that

$$X_{\pi^*f} = X_f^{\mathrm{H}} + (\pi^*f - \xi(f) \circ \pi)E.$$

(ii) *The Jacobi structure* $(\bar{\Lambda}_2, E)$. The Jacobi structure $(\bar{\Lambda}_2, E)$ on $\bar{M}$ is not transitive. In fact, using corollary 3.3 and the fact that the characteristic foliation D of (M, Λ) is the distribution of codimension 1 given by $\eta = 0$, we deduce that the characteristic foliation $\bar{D}$ of the Jacobi manifold $(\bar{M}, \bar{\Lambda}_2, E)$ is the completely integrable distribution defined by $\pi^*\eta = 0$. Moreover, it follows that

$$\bar{D} = D^{\mathrm{H}} \oplus \langle E \rangle \qquad T\bar{M} = \bar{D} \oplus \langle \xi^{\mathrm{H}} \rangle.$$

On the other hand, let $\bar{L}$ be a leaf of the characteristic foliation of $\bar{M}$ and $((\bar{\Lambda}_2)_{\bar{L}}, E_{\bar{L}})$ the transitive Jacobi structure on $\bar{L}$ (see section 2.3). Then, if $\theta_{\bar{L}}$ is the restriction to $\bar{L}$ of the 1-form θ we have that $\theta_{\bar{L}}$ is a contact 1-form on $\bar{L}$ and the Jacobi structure on $\bar{L}$ induced by the 1-form $\theta_{\bar{L}}$ is just $((\bar{\Lambda}_2)_{\bar{L}}, E_{\bar{L}})$. That is, the leaves of this Jacobi structure are all contact manifolds.

Finally, using corollary 3.3, we obtain the relation between the Hamiltonian vector fields of M and $\bar{M}$. This relation is given by

$$X_{\pi^* f} = X_f^{\mathrm{H}} + (\pi^* f)E$$

for $f \in C^{\infty}(M, \mathbb{R})$.

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